Name [redacted], ID [redacted]

Nadeen Tarek

Exam II, MTH 205, Fall 2019

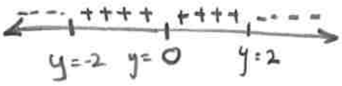
Ayman Badawi

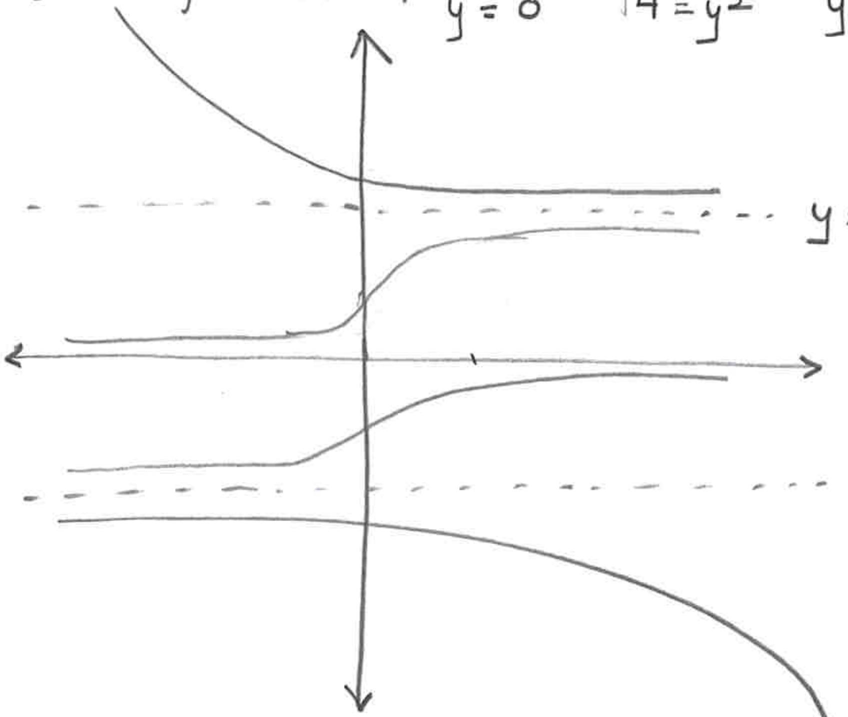
Total = 60 (Excellent) / 60

(Consider Math Major (double major) or minor)

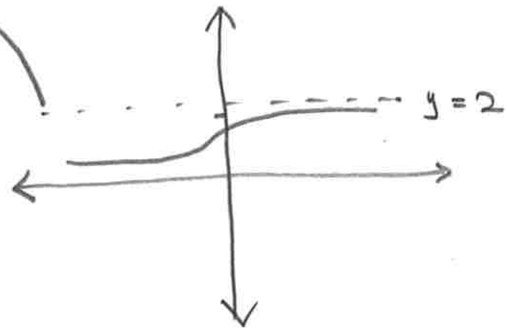
QUESTION 1. (6 points) (1) Given $y' = y^2(4-y^2)$. Find the critical points (values). Sketch all possible solution curves in the region. Classify each critical point

$y=0 \quad \sqrt{4}=y^2 \quad y=\pm 2$





(2) If the point $(1, 1.5)$ lies on the curve, then sketch the solution curve.



QUESTION 2. (6 points) Solve the diff. equation $y' = \frac{e^{2x-y}}{y}$

$$\frac{dy}{dx} = \frac{e^{2x-y}}{y} \qquad e^{2x}\cdot e^{-y}$$

$$\frac{dy}{dx} = \frac{e^{2x}}{ye^y}$$

$$\int ye^y\,dy = \int e^{2x}\,dx$$

$$\boxed{ye^y - e^y = \frac{e^{2x}}{2} + C}$$

$$\int ye^y = ye^y - e^y$$

y	$\int$
1	$(+)\ e^y$
0	$(-)\ e^y$
	e^y

QUESTION 3. (6 points) Solve the diff. equation $y' = \frac{-2xy}{1-x^2}$, where $x \geq 4$

$$y' = \frac{-2xy}{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2xy}{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{1-x^2} \cdot y$$

$$\int \frac{1}{y}\,dy = \int \frac{-2x}{1-x^2}\,dx$$

$$\boxed{\ln|y| = \ln|1-x^2| + C}$$

QUESTION 4. (6 points) Solve the diff. equation $y' = \frac{y\cos(xy)-e^{2y}}{2xe^{2y}-x\cos(xy)+2y}$ [Hint: assume that it is exact, no need to check $F_{xy} = F_{yx}$]

$$y' = \frac{\overbrace{y\cos(xy) - e^{2y}}^{-F_x}}{\underbrace{2xe^{2y} - x\cos(xy) + 2y}_{F_y}}$$

$$[\overbrace{2xe^{2y} - x\cos(xy) + 2y}^{F_y}]\,dy + [\overbrace{-y\cos(xy) + e^{2y}}^{F_x}]\,dx = 0$$

$$\int F_x\,dx = \int -y\cos(xy) + e^{2y}\,dx = -\frac{y}{y}\sin(xy) + e^{2y}x + c(y)$$

$$= -\sin(xy) + x e^{2y} + c(y)$$

$$F_y = -x\cos(xy) + 2xe^{2y} + c'(y) = 2xe^{2y} - x\cos(xy) + 2y$$

$$\int c'(y) = \int 2y\,dy \quad c(y) = y^2 + c$$

$$\boxed{-\sin(xy) + xe^{2y} + y^2 + C = 0}$$

Nadeen

QUESTION 5. (6 points) Imagine a cake is removed from an oven, its temperature is measured 300 F. The cake was placed in a room that has temperature 70 F. Three minutes later its temperature is 200 F. Find the temperature of the cake at any time t. How long will it take for the cake to reach temperature 74 F?

$T(0) = 300 \quad T(3) = 200 \qquad T_m = 70^\circ$

$\frac{dT}{dt} = T' = \alpha (T - T_m)$

$T' = \alpha(T - 70)$

$T' - \alpha T = -70\alpha \qquad I = e^{\int -\alpha\, dt} = e^{-\alpha t}$

$T = \frac{\int e^{-\alpha t} \cdot -70\alpha\, dt}{e^{-\alpha t}} = \frac{\frac{\cancel{-}70\cancel{\alpha}}{\cancel{-\alpha}} e^{-\alpha t} + C}{e^{-\alpha t}} = 70 + Ce^{\alpha t}$

$\boxed{T = 70 + Ce^{\alpha t}}$

$300 = 70 + Ce^0$

$C = 230$

$T = 70 + 230e^{\alpha t}$

$200 = 70 + 230e^{3\alpha}$

$e^{3\alpha} = \frac{200-70}{230}$

$\alpha = \frac{\ln\left(\frac{13}{23}\right)}{3} \qquad \alpha = -0.190$

$\boxed{T = 70 + 230e^{-0.190t}}$

$74 = 70 + 230e^{-0.190t}$

$e^{-0.190t} = \frac{74-70}{230}$

$t = \frac{\ln\left(\frac{2}{115}\right)}{-0.190} = 21.3 \text{ min}$

QUESTION 6. (6 points) Given $(7x+2)y'' - 7y' + (-9-7x)y = 0$. Given $y_1 = e^{-x}$ is a solution. Find y_2, then find the general solution.

$\frac{(7x+2)}{7x+2} y'' - \frac{7y'}{7x+2} + \frac{(-9-7x)}{7x+2} y = 0 \qquad y_1 = e^{-x}$

$y'' - \frac{7}{7x+2} y' + \frac{(-9-7x)}{7x+2} y = 0$

$e^{\int \frac{7}{7x+2} dx} = \cancel{e}^{\ln(7x+2)}$

$= 7x+2$

$y_2 = y_1 \int \frac{e^{\int -Q_x dx}}{y_1^2} dx = e^{-x} \int \frac{7x+2}{e^{-2x}} dx$

$y_2 = e^{-x}\left[\frac{1}{2}(7x+2)e^{2x} - \frac{7}{4}e^{2x}\right]$

$y_2 = \frac{1}{2}(7x+2)e^{x} - \frac{7}{4}e^{x}$

$y_g = C_1 e^{-x} + C_2\left[\frac{1}{2}(7x+2)e^{x} - \frac{7}{4}e^{x}\right]$

$\int (7x+2)e^{2x}$

$7x+2$	$\int e^{2x}$
7	$\oplus \frac{e^{2x}}{2}$
0	$\ominus \frac{e^{2x}}{4}$

$\frac{1}{2}(7x+2)e^{2x} - \frac{7}{4}e^{2x}$

$+C$

$m(m-1)-3m+3$
$m^2-m-3m+3$
m^2-4m+3

QUESTION 7. (10 points) (1) Solve for y, $x^2y'' - 3xy' + 3y = 2x^4e^x$ [Hint : $y = y_h + y_p$]

$x^2y'' - 3xy' + 3y = 2x^4e^x \quad y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$\cancel{x^2}(m(m-1)x^{m-2}) - (3\cancel{x}(mx^{m-1})) + 3(x^m) = 0$

$x^m[m^2 - m - 3m + 3] = 0$

$m^2 - 4m + 3 = 0$

$m = 3 \quad m = 1$

$\boxed{y_h = C_1 x^3 + C_2 x}$

$y_1 = x^3, \quad y_2 = x$

$y_p = v_1y_1 + v_2y_2$

$v_1'y_1 + v_2'y_2 = 0$

$v_1'y_1' + v_2'y_2' = \frac{2x^4e^x}{\cancel{x^2}}$

$v_1'x^3 + v_2'x = 0$

$v_1'(3x^2) + v_2'(1) = 2x^2e^x$

$W = \begin{vmatrix} x^3 & x \\ 3x^2 & 1 \end{vmatrix} = x^3 - 3x^3 = -2x^3$

$v_1' = \frac{\begin{vmatrix} 0 & x \\ 2x^2e^x & 1 \end{vmatrix}}{-2x^3} = \frac{\cancel{-2x^3}e^x}{\cancel{-2x^3}} = e^x$

$\boxed{v_1 = e^x}$

$v_2' = \frac{\begin{vmatrix} x^3 & 0 \\ 3x^2 & 2x^2e^x \end{vmatrix}}{-2x^3} = \frac{\cancel{2}x^5e^x}{-\cancel{2}\cancel{x^3}} = -x^2e^x$

$v_2 = \int -x^2e^x$

	$\int$
$-x^2$	e^x (+)
$-2x$	e^x (−)
-2	e^x (+)
0	e^x

$\boxed{v_2 = -x^2e^x + 2xe^x - 2e^x}$

$y_p = x^3e^x + (-x^2e^x + 2xe^x - 2e^x)x$

$y_p = \cancel{x^3e^x} - \cancel{x^3e^x} + 2x^2e^x - 2xe^x$

$\boxed{y_p = 2xe^x(x-1)}$

$\boxed{y = C_1x^3 + C_2x + 2xe^x(x-1)}$

(2) $(\sqrt{2x+6})y' + \frac{1}{x-4}y = \frac{1}{x-6}$, where $y(1) = 7$. Find the largest interval for the values of x so that the solution is unique.

$\sqrt{2x+6}\,y' + \frac{1}{x-4}y = \frac{1}{x-6}$

$2x + 6 = 0$

$x = \frac{-6}{2}$

$\boxed{x = -3}$

$x \neq -3$

$x \leq -3$

$x \neq 4$

interval $(-3, 4)$

-3 4

Nadeen

QUESTION 8. (6 points) Solve the diff. equation $\frac{dy}{dx} = \frac{1}{-2x+y^2+1}$

$$\frac{dy}{dx} = \frac{1}{-2x+y^2+1}$$

$$\frac{dx}{dy} = -2x+y^2+1$$

$$x' = -2x+y^2+1$$

$$x'+2x = y^2+1 \qquad I = e^{\int 2\,dy} = e^{2y}$$

$$x = \frac{\int e^{2y}\cdot(y^2+1)\,dy}{e^{2y}}$$

$$x = \frac{\int y^2e^{2y}+e^{2y}}{e^{2y}} = \frac{\cancel{\frac{1}{2}y^2e^{2y}-\frac{1}{2}ye^{2y}+\frac{3}{4}e^{2y}}+C}{e^{2y}}$$

$$\boxed{x = \frac{1}{2}y^2 - \frac{1}{2}y + \frac{3}{4} + Ce^{-2y}}$$

$$y^2e^{2y} + e^{2y}$$

$$\int y^2 e^{2y} + e^{2y}$$

$$\frac{e^{2y}}{2}$$

	$\int$
y^2	$(+)\ e^{2y}$
$2y$	$(-)\ \frac{1}{2}e^{2y}$
2	$(+)\ \frac{1}{4}e^{2y}$
0	$\frac{1}{8}e^{2y}$

$$\frac{1}{2}y^2e^{2y} - \frac{1}{2}ye^{2y} + \frac{1}{4}e^{2y} + \frac{1}{2}e^{2y}$$

$$= \frac{1}{2}y^2e^{2y} - \frac{1}{2}ye^{2y} + \frac{3}{4}e^{2y} + C$$

QUESTION 9. (8 points) Imagine a company sells fake-honey. A tank contains 200 liters of fluid in which 30 grams of honey is dissolved (i.e, $A(0) = 30$). Brine containing 3 grams of honey per liter is then pumped into the tank at rate 4L/min. The well-mixed solution is pumped out at 6L/min. Find the number $A(t)$ of grams of honey in the tank at time t. When is the tank empty?

$$A' = \text{In} - \text{out}$$

$$A' = (3)(4) - C(t)(6)$$

$$C(t) = \frac{A(t)}{200+(4-6)t}$$

$$A' = 12 - \frac{6A(t)/2}{200-2t\ /2}$$

$$A' + \frac{3A(t)}{100-t} = 12 \qquad I = e^{\int \frac{3}{100-t}dt} = e^{-3\ln(100-t)} = (100-t)^{-3}$$

$$A = \frac{\int (100-t)^{-3}\cdot 12\,dt}{(100-t)^{-3}}$$

$$\int 12(100-t)^{-3}\,dt$$

$$12\int (100-t)^{-3}\,dt$$

$$= +\frac{12}{2} * -1\,(100-t)^{-2}$$

$$= 6(100-t)^{-2} + C$$

$$A = \frac{6(100-t)^{-2}+C}{(100-t)^{-3}}$$

$$\boxed{A = 6(100-t) + C(100-t)^3}$$

$$30 = 6(100-0) + C(100-0)^3$$

$$30 = 600 + (100)^3C$$

$$C = \frac{-57}{1\times 10^5}$$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

$$200+(4-6)t = 0$$

$$-2t = -200$$

$$t = \frac{+200}{+2}$$

$t = 100$ min the tank is empty

$$\boxed{A = 6(100-t) - \frac{57}{10^5}(100-t)^3}$$